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CONVERGENCE OF AN ENERGY-PRESERVING SCHEME FOR THE ZAKHAROV EQUATIONS IN ONE SPACE DIMENSION

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ABSTRACT. An energy-preserving, linearly implicit finite difference scheme is presented for approximating solutions to the periodic Cauchy problem for the one-dimensional Zakharov system of two nonlinear partial differential equations. First-order convergence estimates are obtained in a standard "energy" norm in terms of the initial errors and the usual discretization errors.

1. INTRODUCTION

In [11] Zakharov introduced a system of equations to model the propagation of Langmuir waves in a plasma. If we denote by $N(x,t)$ $(x \in \mathbb{R}, t > 0)$ the deviation of the ion density from its equilibrium value, and by $E(x,t)$ the envelope of the high-frequency electric field, then the one-dimensional system takes the form

$$\text{(ZS.}E\text{)} \qquad iE_t + E_{xx} = NE,$$

$$\text{(ZS.}N\text{)} \qquad N_{tt} - N_{xx} = \frac{\partial^2}{\partial x^2}(|E|^2).$$

We solve on $\{x \in \mathbb{R}, t > 0\}$ and supplement (ZS) by prescribing initial values for E, N, and N_t:

$$(1) \qquad E(x,0) = E^0(x), \quad N(x,0) = N^0(x), \quad N_t(x,0) = N^1(x).$$

Most of the interest to date in (ZS) stems from two particular features. Firstly, (ZS) admits solitary wave solutions [3]. Secondly, in three space dimensions, (ZS) was derived to model the collapse of caverns (cf. [11]). An intriguing and still unresolved question remains in three dimensions as to whether smooth data can generate a solution which becomes singular in *finite* time.

As is well known, (ZS) possesses the two formal invariants

$$(2) \qquad \int_{-\infty}^{\infty} |E(x,t)|^2\,dx = \int_{-\infty}^{\infty} |E(x,0)|^2\,dx,$$

$$(3) \qquad \int_{-\infty}^{\infty} \left(|E_x|^2 + \tfrac{1}{2}(|v|^2 + N^2) + N|E|^2\right) dx = \text{const},$$

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where v is given by

$$v = -u_x, \qquad u_{xx} = N_t. \tag{4}$$

We know that these are sufficient for global weak existence (cf. [9]). Also from [9] the same conclusion holds in three dimensions under an additional "smallness" condition. Moreover, higher-order estimates from [9] guarantee the existence of a *smooth solution* in one dimension provided smooth data are prescribed.

It is such a smooth solution of (ZS) with periodic boundary conditions which we approximate numerically in this paper. A spectral method is used in [5]; while practical results seem very good, the convergence issue is not rigorously addressed. Our algorithm uses an approximation of "Crank-Nicolson" type on the linear parts of (ZS). We approximate the solution over a fixed but arbitrary time interval $0 \le t \le T$.

The nonlinear terms in (ZS) are then approximated in such a way that:

(i) the discrete L^2-norm (over a period) of the approximation to E is conserved; and

(ii) a discrete analogue of the total energy is conserved.

This discrete energy will be shown to be bounded below by a positive definite form. The scheme is linearly implicit and involves only two periodic tridiagonal solvers to advance one step in time. We obtain first-order convergence estimates in the natural "energy norm" in terms of initial errors and standard discretization errors.

In the references we list several papers where conservative schemes have been employed [2, 4, 6, 8]. Related results are to be found in [1, 10].

The standard summation by parts formula is

$$\sum_{j=1}^{J} v_j(u_{j+1} - 2u_j + u_{j-1}) = v_{J+1}(u_{J+1} - u_J) - v_1(u_1 - u_0) - \sum_{j=1}^{J}(v_{j+1} - v_j)(u_{j+1} - u_j).$$

The "summed" terms cancel whenever $\{u_k\}, \{v_k\}$ are J-periodic mesh functions.

Although [9] treats the Cauchy problem on all of space, the methods given there (i.e., Galerkin) could be extended to deal with the periodic case studied here. Constants depending on T and the Cauchy data are written c_T, while constants depending only on the data are generically written as c. These will change from line to line without explicit mention.

This scheme has been implemented; details will appear elsewhere.

2. The finite difference scheme

Let $T > 0$ be arbitrary; we will approximate the solution to the periodic Cauchy problem for (ZS) over the time interval $0 \le t \le T$. We first state hypotheses on the Cauchy data and the solution:

(H0) The Cauchy data

$$E(x,0) = E^0(x), \quad N(x,0) = N^0(x), \quad N_t(x,0) = N^1(x)$$

are C^∞ and L-periodic. Moreover,

$$\int_0^L N^1(x)\,dx = 0,$$

$$\sum_{j=1}^{J} N^1(jh) = 0 \quad \text{for any } h > 0 \text{ with } Jh = L.$$

(HE) The periodic Cauchy problem possesses a unique smooth global solution.

In order to write the scheme, we define

$$\delta u_k \equiv \Delta x^{-1}(u_{k+1} - u_k), \tag{5'}$$

$$\delta^2 u_k \equiv \Delta x^{-2}(u_{k+1} - 2u_k + u_{k-1}), \tag{5''}$$

$$\lambda = \frac{\Delta t}{\Delta x}, \qquad \beta = \frac{\Delta t}{\Delta x^2} \tag{6}$$

with $\Delta t, \Delta x > 0$. Now for J a positive integer we choose $\Delta x = \frac{L}{J}$, $\Delta t > 0$ such that

$$n\Delta t \le T \tag{7}$$

and define $t^l = l\Delta t$, $x_j = j\Delta x$ $\quad (l = 0, \dots, n; \quad j = 0, \dots, J)$.

Our scheme is

$$i\frac{E_k^{n+1} - E_k^n}{\Delta t} + \frac{1}{2}\delta^2 E_k^n + \frac{1}{2}\delta^2 E_k^{n+1} = \frac{1}{4}(N_k^n + N_k^{n+1})(E_k^n + E_k^{n+1}), \tag{8.E}$$

$$\frac{N_k^{n+1} - 2N_k^n + N_k^{n-1}}{\Delta t^2} - \frac{1}{2}\delta^2 N_k^{n+1} - \frac{1}{2}\delta^2 N_k^{n-1} = \delta^2(|E_k^n|^2). \tag{8.N}$$

In both relations $k = 1, \dots, J$, $\;n \ge 0$ in the first and $n \ge 1$ in the second. Here we take E_k^n, N_k^n to be J-periodic mesh functions, i.e.,

$$E_k^n = E_j^n, \quad N_k^n = N_j^n \quad \text{if } k \equiv j \pmod{J}.$$

The scheme is supplemented with the initial values

$$E_k^0 = E^0(x_k), \tag{9}$$

$$N_k^0 = N^0(x_k), \qquad N_k^1 = N_k^0 + \Delta t N^1(x_k). \tag{10}$$

We claim that the scheme is uniquely solvable: multiplying (8.N) by Δt^2, we see that the coefficient matrix for the unknown $\{N_k^{n+1}\}_{k=1}^J$, of order $J \times J$, is

$$A_N = \begin{bmatrix} 1+\lambda^2 & -\frac{\lambda^2}{2} & 0 & \cdots & -\frac{\lambda^2}{2} \\ -\frac{\lambda^2}{2} & 1+\lambda^2 & -\frac{\lambda^2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\lambda^2}{2} & 0 & \cdots & -\frac{\lambda^2}{2} & 1+\lambda^2 \end{bmatrix}, \tag{11}$$

which is invertible by Gerschgorin for any $\lambda > 0$. The coefficient matrix for the unknown $\{E_k^{n+1}\}_{k=1}^J$ has the form

$$iI - A_E, \tag{12}$$

where both matrices are square and of order $J \times J$.

A_E is symmetric and has the form

$$\text{(13)} \qquad A_E = \begin{pmatrix} (A_E)_{11} & -\frac{\beta}{2} & 0 & \dots & -\frac{\beta}{2} \\ -\frac{\beta}{2} & (A_E)_{22} & -\frac{\beta}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{\beta}{2} & 0 & \dots & -\frac{\beta}{2} & (A_E)_{JJ} \end{pmatrix},$$

where

$$\text{(14)} \qquad (A_E)_{kk} = \beta + \frac{\Delta t}{4}(N_k^n + N_k^{n+1}).$$

Since A_E has only real eigenvalues, $iI - A_E$ is invertible. Thus the scheme is uniquely solvable at each time step. Indeed, putting $n = 0$ in $(8.E)$, we can solve for $\{E_k^1\}$, since N_k^0, N_k^1, E_k^0 are known from the data. Putting $n = 1$ in $(8.N)$, we can then solve for $\{N_k^2\}$ and, using $\{N_k^2\}$, we can put $n = 1$ in $(8.E)$ and solve for $\{E_k^2\}$, etc.

We summarize with

Lemma 1. *Assume the data satisfy* (H0). *Then the scheme* (8.*E*), (8.*N*) *is uniquely solvable at each time step.*

Lemma 2. *Let the data satisfy* (H0). *Define* $\{u_k^n\}$ *by*

$$\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} = \frac{N_k^{n+1} - N_k^n}{\Delta t}, \qquad k = 1, \dots, J-1,$$

$$u_0 = u_J = 0.$$

Extend $\{u_k^n\}$ *by defining*

$$u_k^n = u_j^n \quad \text{if} \quad k \equiv j \pmod J.$$

Then

$$u_k^n = -\Delta x \sum_{j=1}^{J-1} G(x_k, x_j) \frac{N_j^{n+1} - N_j^n}{\Delta t},$$

where

$$G(x, y) = \begin{cases} x(1 - \frac{y}{L}), & 0 \le x \le y \le L, \\ y(1 - \frac{x}{L}), & 0 \le y \le x \le L. \end{cases}$$

Proof. The proof that the given representation is indeed a solution is a straightforward computation and is omitted. The only issue is one of compatibility. Summing the definition of u_k^n, we see that it is required that

$$\sum_{k=1}^{J} (N_k^{n+1} - N_k^n) = 0.$$

When $n = 0$, this is true by hypotheses (H0) and (10). Using $(8.N)$, we can write

$$N_k^{n+1} - N_k^n = N_k^n - N_k^{n-1} + \frac{\Delta t^2}{2}\delta^2(N_k^{n+1} + N_k^{n-1} + 2|E_k^n|^2).$$

Using induction, we sum both sides over k. The sum of the first two terms on the right vanishes by the induction hypothesis; the sum of the remaining terms vanishes by periodicity. □

Theorem 1. *Let the data satisfy* (H0). *Then the scheme* (8) *possesses the following two invariants:*

(a)

$$\sum_k |E_k^n|^2 \Delta x = \text{const} \qquad (n\Delta t \le T).$$

(b) *Define* u_k^n *as in Lemma 2, so that* $\delta^2 u_k^n = (N_k^{n+1} - N_k^n)/\Delta t$. *Then*

$$\mathscr{E}_d^{n+1} \equiv \Delta x \sum_k \Big[|\delta E_k^{n+1}|^2 + \tfrac{1}{2}(\delta u_k^n)^2 + \tfrac{1}{4}\{(N_k^n)^2 + (N_k^{n+1})^2\} + \tfrac{1}{2}(N_k^n + N_k^{n+1})|E_k^{n+1}|^2 \Big] = \text{const}$$

for $n\Delta t \le T$. *The sums run over* $1 \le k \le J$.

Thus the discrete L^2-norm of E^n over a period is conserved, and the form of $\mathscr{E}_d^n$ is similar to that for the exact solution in (2), (3).

We show that $\mathscr{E}_d^n$ is bounded below by a positive definite form. For this purpose, we put

$$\|E^n\|_2^2 \equiv \sum_k |E_k^n|^2 \Delta x, \tag{15}$$

$$\|\delta E^n\|_2^2 \equiv \sum_k |\delta E_k^n|^2 \Delta x, \tag{16}$$

with similar quantities for N^n. We make note of the discrete Sobolev inequality

$$\sup_k |u_k| \le c\|u\|_2^{1/2} \|\delta u\|_2^{1/2} \tag{17}$$

valid for periodic mesh functions $\{u_k\}$. Indeed, denoting the Fourier coefficients of the mesh function u by $\{c_m\}$, we write

$$\begin{aligned} |u_k| &\le c \left(\sum_{|m|\le M} + \sum_{|m|>M} \right) |c_m| \\ &\le cM^{1/2} \left(\sum_m |c_m|^2 \right)^{1/2} + cM^{-(1/2)} \left(\sum_m |m|^2 |c_m|^2 \right)^{1/2} \end{aligned}$$

and optimize on M.

The last term $\mathscr{L}$ in $\mathscr{E}_d^n$ is estimable by

$$\begin{aligned} |\mathscr{L}| &\le \frac{1}{2} \sum_k |N_k^n| |E_k^{n+1}|^2 \Delta x + \frac{1}{2} \sum_k |N_k^{n+1}| |E_k^{n+1}|^2 \Delta x \\ &\le \frac{\varepsilon}{4} \sum_k ((N_k^n)^2 + (N_k^{n+1})^2) \Delta x + \frac{1}{2\varepsilon} \sum_k |E_k^{n+1}|^4 \Delta x \end{aligned}$$

for any $\varepsilon > 0$. Choosing $\varepsilon = \frac{1}{2}$, we get the bound

$$|\mathscr{L}| \le \frac{1}{8} \sum_k \Delta x ((N_k^n)^2 + (N_k^{n+1})^2) + \|E^{n+1}\|_4^4.$$

By the Sobolev inequality (17) and part (a) of the theorem,

$$\|E^{n+1}\|_4^4 \le c\|E^{n+1}\|_2^2\|E^{n+1}\|_\infty^2 \le c\|E^{n+1}\|_\infty^2 \le c\|\delta E^{n+1}\|_2 \le \frac{1}{4}\|\delta E^{n+1}\|_2^2 + c.$$

This gives us

Lemma 3. *There is a constant* c, *depending only on the data, such that the solution of the discrete scheme* (8.E), (8.N) *satisfies*

$$\sum_k \Delta x\left[|E_k^{n+1}|^2 + |\delta E_k^{n+1}|^2 + (\delta u_k^n)^2 + (N_k^n)^2 + (N_k^{n+1})^2\right] \le c,$$

and hence $\sup_k |E_k^n| \le c$.

Proof of Theorem 1. As is well known, part (a) is obtained by multiplying (8.E) by $\overline{E}_k^{n+1} + \overline{E}_k^n$, summing over k, $k = 1, \dots, J$, and taking the imaginary part.

In order to verify (b), we multiply (8.E) by $\overline{E}_k^{n+1} - \overline{E}_k^n$ and sum on k. Adding this to its conjugate, we obtain

$$I_n + I_{n+1} = \frac{1}{4}\sum_k (N_k^{n+1} + N_k^n) \cdot 2\,\mathrm{Re}(E_k^{n+1} + E_k^n)(\overline{E}_k^{n+1} - \overline{E}_k^n), \tag{18}$$

where

$$I_m = \frac{1}{\Delta x^2}\,\mathrm{Re}\sum_k (\overline{E}_k^{n+1} - \overline{E}_k^n)(E_{k+1}^m - 2E_k^m + E_{k-1}^m) \qquad (m = n, n+1).$$

The right side of (18) equals

$$\frac{1}{2}\sum_k (|E_k^{n+1}|^2 - |E_k^n|^2)(N_k^{n+1} + N_k^n). \tag{19}$$

Summing by parts, we get for the left side of (18)

$$I_n + I_{n+1} = -\frac{1}{\Delta x^2}\sum_k |E_{k+1}^{n+1} - E_k^{n+1}|^2 + \frac{1}{\Delta x^2}\sum_k |E_{k+1}^n - E_k^n|^2. \tag{20}$$

Thus (19), (20) yield the identity

$$-\sum_k |\delta E_k^{n+1}|^2 + \sum_k |\delta E_k^n|^2 = \frac{1}{2}\sum_k (|E_k^{n+1}|^2 - |E_k^n|^2)(N_k^{n+1} + N_k^n). \tag{21}$$

We obtain the contribution from $\{N_k^n\}$ by recalling from Lemma 2 that

$$\delta^2 u_k^n \equiv \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} = \frac{N_k^{n+1} - N_k^n}{\Delta t} \tag{22}$$

and by multiplying (8.N) by $\frac{1}{2}(u_k^n + u_k^{n-1})$ and then summing on k. There results

$$\mathrm{I} - \mathrm{II} = \mathrm{III}, \tag{23}$$

where

$$\mathrm{I} = \frac{1}{2}\sum_k \frac{(N_k^{n+1} - 2N_k^n + N_k^{n-1})}{\Delta t^2}(u_k^n + u_k^{n-1}),$$

$$\mathrm{II} = \frac{1}{4}\sum_k \frac{(u_k^n + u_k^{n-1})}{\Delta x^2}[N_{k+1}^{n+1} - 2N_k^{n+1} + N_{k-1}^{n+1} + N_{k+1}^{n-1} - 2N_k^{n-1} + N_{k-1}^{n-1}],$$

$$\mathrm{III} = \frac{1}{2}\sum_k \frac{(u_k^n + u_k^{n-1})}{\Delta x^2}[|E_{k+1}^n|^2 - 2|E_k^n|^2 + |E_{k-1}^n|^2].$$

Term III is summed by parts:

$$\begin{aligned}\mathrm{III} &= -\frac{1}{2\Delta x^2}\sum_k [(u_{k+1}^n + u_{k+1}^{n-1}) - (u_k^n + u_k^{n-1})][|E_{k+1}^n|^2 - |E_k^n|^2] \\ &= -\frac{1}{2\Delta x^2}\sum_k [u_k^n + u_k^{n-1} - u_{k-1}^n - u_{k-1}^{n-1}]|E_k^n|^2 \\ &\quad + \frac{1}{2\Delta x^2}\sum_k [u_{k+1}^n + u_{k+1}^{n-1} - u_k^n - u_k^{n-1}]|E_k^n|^2,\end{aligned} \tag{24}$$

where we have shifted $k \to k-1$ to obtain the first sum. Thus, by (22),

$$\begin{aligned}\mathrm{III} &= \frac{1}{2\Delta x^2}\sum_k |E_k^n|^2[(u_{k+1}^n - 2u_k^n + u_{k-1}^n) + (u_{k+1}^{n-1} - 2u_k^{n-1} + u_{k-1}^{n-1})] \\ &= \frac{1}{2}\sum_k |E_k^n|^2\left[\frac{N_k^{n+1} - N_k^n}{\Delta t} + \frac{N_k^n - N_k^{n-1}}{\Delta t}\right] \\ &= \frac{1}{2\Delta t}\sum_k |E_k^n|^2(N_k^{n+1} - N_k^{n-1}).\end{aligned} \tag{25}$$

To evaluate I, we note that by (22)

$$\delta^2 u_k^n - \delta^2 u_k^{n-1} = \frac{N_k^{n+1} - N_k^n}{\Delta t} - \left(\frac{N_k^n - N_k^{n-1}}{\Delta t}\right) = \frac{N_k^{n+1} - 2N_k^n + N_k^{n-1}}{\Delta t}.$$

Thus,

$$\mathrm{I} = \frac{1}{2\Delta t}\sum_k (u_k^n + u_k^{n-1})[\delta^2 u_k^n - \delta^2 u_k^{n-1}]$$

and, summing this by parts, we get

$$\mathrm{I} = -\frac{1}{2\Delta t}\sum_k (\delta u_k^n)^2 + \frac{1}{2\Delta t}\sum_k (\delta u_k^{n-1})^2. \tag{26}$$

Summing II now by parts, we find

$$\begin{aligned}\mathrm{II} = &-\frac{1}{4\Delta x^2}\sum_k [(u_{k+1}^n + u_{k+1}^{n-1}) - (u_k^n + u_k^{n-1})] \\ &\cdot [(N_{k+1}^{n+1} - N_k^{n+1}) + (N_{k+1}^{n-1} - N_k^{n-1})] \\ = &-\frac{1}{4\Delta x^2}\sum_k [u_k^n + u_k^{n-1} - u_{k-1}^n - u_{k-1}^{n-1}][N_k^{n+1} + N_k^{n-1}] \\ &+ \frac{1}{4\Delta x^2}\sum_k [u_{k+1}^n + u_{k+1}^{n-1} - u_k^n - u_k^{n-1}][N_k^{n+1} + N_k^{n-1}],\end{aligned}$$

where we have again shifted $k \to k-1$ to get the first sum. Thus, by (22),

$$\begin{aligned}\mathrm{II} &= \frac{1}{4\Delta x^2}\sum_k (N_k^{n+1} + N_k^{n-1})[(u_{k+1}^n - 2u_k^n + u_{k-1}^n) + (u_{k+1}^{n-1} - 2u_k^{n-1} + u_{k-1}^{n-1})] \\ &= \frac{1}{4}\sum_k (N_k^{n+1} + N_k^{n-1})\left[\frac{N_k^{n+1} - N_k^n}{\Delta t} + \frac{N_k^n - N_k^{n-1}}{\Delta t}\right] \\ &= \frac{1}{4\Delta t}\sum_k [(N_k^{n+1})^2 - (N_k^{n-1})^2].\end{aligned}$$

Therefore, equation (23) yields

$$\begin{aligned}&-\frac{1}{2\Delta t}\sum_k (\delta u_k^n)^2 - \frac{1}{4\Delta t}\sum_k (N_k^{n+1})^2 \\ &\quad = -\frac{1}{2\Delta t}\sum_k (\delta u_k^{n-1})^2 - \frac{1}{4\Delta t}\sum_k (N_k^{n-1})^2 \\ &\qquad + \frac{1}{2\Delta t}\sum_k |E_k^n|^2 (N_k^{n+1} - N_k^{n-1}).\end{aligned} \tag{27}$$

Now multiply this by Δt and add the result to (21) to get

$$\begin{aligned}&-\frac{1}{2}\sum_k (\delta u_k^n)^2 - \frac{1}{4}\sum_k (N_k^{n+1})^2 - \sum_k |\delta E_k^{n+1}|^2 \\ &\quad = -\frac{1}{2}\sum_k (\delta u_k^{n-1})^2 - \frac{1}{4}\sum_k (N_k^{n-1})^2 - \sum_k |\delta E_k^n|^2 \\ &\qquad + \frac{1}{2}\sum_k [|E_k^n|^2 (N_k^{n+1} - N_k^{n-1}) + (|E_k^{n+1}|^2 - |E_k^n|^2)(N_k^{n+1} + N_k^n)].\end{aligned} \tag{28}$$

The last term here equals

$$\frac{1}{2}\sum_k |E_k^{n+1}|^2 (N_k^{n+1} + N_k^n) - \frac{1}{2}\sum_k |E_k^n|^2 (N_k^n + N_k^{n-1}).$$

Therefore, when we define $\mathscr{E}_d^{n+1}$ as in part (b) of Theorem 1, (28) implies $\mathscr{E}_d^{n+1} = \mathscr{E}_d^n$ and hence $\mathscr{E}_d^n = \mathscr{E}_d^0$ and energy is conserved. □

In order to state the main theorem, we define the *errors* by

$$e_k^n = E(x_k, t^n) - E_k^n, \tag{29}$$

$$\eta_k^n = N(x_k, t^n) - N_k^n. \tag{30}$$

Here, E_k^n, N_k^n are computed from the scheme (8.E), (8.N) for $n\Delta t \le T$, $1 \le k \le J$.

Lemma 4. *Let the data satisfy* (H0). *Define* $\{U_k^n\}$ *by*

$$\frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{\Delta x^2} = \frac{\eta_k^{n+1} - \eta_k^n}{\Delta t}, \qquad k = 1, \dots, J-1, \tag{31}$$
$$U_0 = U_J = 0.$$

Extend $\{U_k^n\}$ *by defining*

$$U_k^n = U_j^n \quad \textit{if} \quad k \equiv j \bmod J.$$

Then

$$U_k^n = -\Delta x \sum_{j=1}^{J-1} G(x_k, x_j) \frac{\eta_j^{n+1} - \eta_j^n}{\Delta t},$$

where

$$G(x, y) = \begin{cases} x(1 - \frac{y}{L}), & 0 \le x \le y \le L, \\ y(1 - \frac{x}{L}), & 0 \le y \le x \le L. \end{cases} \tag{32}$$

Proof. The actual computation showing that the given representation is a solution is easy and is omitted. As in Lemma 2, there remains the compatibility question. Using the definition (30) of η_k^n, we have

$$\begin{aligned} \delta^2 U_k^n &= \Delta t^{-1}[N(x_k, t^{n+1}) - N_k^{n+1} - N(x_k, t^n) + N_k^n] \\ &= -\delta^2 u_k^n + \Delta t^{-1}[N(x_k, t^{n+1}) - N(x_k, t^n)]. \end{aligned}$$

Therefore, as in Lemma 2, we require that

$$S \equiv \sum_{k=1}^{J} [N(x_k, t^{n+1}) - N(x_k, t^n)] = 0.$$

We expand $N(x, t)$ in a Fourier series with Fourier coefficients $\{c_m\}$:

$$N(x, t) = \sum_m c_m(t) \exp\left(\frac{2im\pi x}{L}\right).$$

Thus, $c_0(t)$ is proportional to $\int_0^L N(x, t)\,dx$. Integrating (ZS.N) over a period, we see that this integral is a linear function of t. In fact, $c_0(t)$ is *constant* in time in view of (H0). Now we write

$$\sum_{k=1}^{J} N(x_k, t) = \sum_m c_m(t) \sum_{k=1}^{J} \exp\left(\frac{2im\pi x_k}{L}\right)$$

and evaluate the inner sum explicitly. Using $x_k = k\Delta x = kL/J$, we see that this sum over k vanishes unless $m = 0$, in which case

$$\sum_{k=1}^{J} N(x_k, t) = Jc_0(t).$$

Hence $S = 0$ as desired. □

The norms are defined, e.g., as $\|e^n\|_2^2 = \sum_{k=1}^{J} |e_k^n|^2 \Delta x$, etc.

Theorem 2. *Let $T > 0$; assume* (HE) *and that the data satisfy* (H0). *Given any positive integer J, let $J\Delta x = L$ and choose $\Delta t = \Delta x$. Let E_k^n, N_k^n be computed from the scheme* (8.*E*), (8.*N*), (9), (10) *for $n\Delta t \leq T$. Define*

$$\mathscr{E}^n = \tfrac{1}{2}\left[\|e^{n+1}\|_2^2 + \|\delta e^{n+1}\|_2^2 + \|\delta U^n\|_2^2 + \tfrac{1}{2}(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2)\right]. \tag{33}$$

(*Thus, $\mathscr{E}^n$ is the* (*square of the*) *"energy norm" of the errors.*)

Then there exists a constant c_T depending only on the data and T, with the property that for Δx sufficiently small, we have

$$\mathscr{E}^n \leq c_T\left[\mathscr{E}^0 + \Delta x^2\right].$$

Moreover, $\mathscr{E}^0 = O(\Delta x^2)$, and hence

$$\mathscr{E}^n \leq c_T\Delta x^2 \quad \textit{as } \Delta x \to 0.$$

The proof of Theorem 2 will be given in the next section.

Remark. The choice $\Delta t = \Delta x$ allows us to easily combine several estimates. It is seen from the proof that the same estimates can be obtained provided Δt is bounded both above and below by a constant times Δx.

3. Convergence estimates, proof of the main theorem

We begin by defining the standard discretization errors

$$\begin{aligned}\tau_k^n = {} & \frac{i}{\Delta t}(E(x_k, t^{n+1}) - E(x_k, t^n)) \\ & + \frac{1}{2\Delta x^2}(E(x_{k+1}, t^n) - 2E(x_k, t^n) + E(x_{k-1}, t^n)) \\ & + \frac{1}{2\Delta x^2}(E(x_{k+1}, t^{n+1}) - 2E(x_k, t^{n+1}) + E(x_{k-1}, t^{n+1})) \\ & - \frac{1}{4}(N(x_k, t^n) + N(x_k, t^{n+1}))(E(x_k, t^n) + E(x_k, t^{n+1}))\end{aligned} \tag{34}$$

and

$$\begin{aligned}\sigma_k^n = {} & \frac{1}{\Delta t^2}(N(x_k, t^{n+1}) - 2N(x_k, t^n) + N(x_k, t^{n-1})) \\ & - \frac{1}{2\Delta x^2}(N(x_{k+1}, t^{n+1}) - 2N(x_k, t^{n+1}) + N(x_{k-1}, t^{n+1})) \\ & - \frac{1}{2\Delta x^2}(N(x_{k+1}, t^{n-1}) - 2N(x_k, t^{n-1}) + N(x_{k-1}, t^{n-1})) \\ & - \frac{1}{\Delta x^2}(|E(x_{k+1}, t^n)|^2 - 2|E(x_k, t^n)|^2 + |E(x_{k-1}, t^n)|^2).\end{aligned} \tag{35}$$

As usual, these measure the amount by which the exact solutions fail to satisfy the approximate equations.

Recall that E, N are smooth solutions.

Lemma 5. *We have $|\tau_k^n| + |\sigma_k^n| = O(\Delta t^2 + \Delta x^2)$ as $\Delta x, \Delta t \to 0$.*

Proof. By Taylor's theorem and (ZS.*E*) we can write the first three terms τ_3

in τ_k^n as

$$\begin{aligned}
\tau_3 &= i\left(E_t(x_k, t^n) + \frac{1}{2}\Delta t E_{tt}(x_k, \beta_k^n)\right) + \frac{1}{2}(E_{xx}(x_k, t^n) + O(\Delta x^2)) \\
&\quad + \frac{1}{2}(E_{xx}(x_k, t^{n+1}) + O(\Delta x^2)) \qquad (t^n < \beta_k^n < t^{n+1}) \\
&= iE_t(x_k, t^n) + \frac{i\Delta t}{2}E_{tt}(x_k, \beta_k^n) + O(\Delta x^2) \\
&\quad + \frac{1}{2}[N(x_k, t^n)E(x_k, t^n) - iE_t(x_k, t^n)] \\
&\quad + \frac{1}{2}[N(x_k, t^{n+1})E(x_k, t^{n+1}) - iE_t(x_k, t^{n+1})] \\
&= \frac{N(x_k, t^n)E(x_k, t^n) + N(x_k, t^{n+1})E(x_k, t^{n+1})}{2} + O(\Delta x^2) \\
&\quad + \frac{i\Delta t}{2}E_{tt}(x_k, \beta_k^n) + \frac{i}{2}[E_t(x_k, t^n) - E_t(x_k, t^{n+1})] \\
&= \frac{N(x_k, t^n)E(x_k, t^n) + N(x_k, t^{n+1})E(x_k, t^{n+1})}{2} + O(\Delta t^2 + \Delta x^2).
\end{aligned}$$

Now the result for τ_k^n will follow if

$$\begin{aligned}
&\tfrac{1}{2}(N(x_k, t^n)E(x_k, t^n) + N(x_k, t^{n+1})E(x_k, t^{n+1})) \\
&\quad - \tfrac{1}{4}(N(x_k, t^n) + N(x_k, t^{n+1}))(E(x_k, t^n) + E(x_k, t^{n+1})) \\
&= O(\Delta t^2 + \Delta x^2).
\end{aligned}$$

Simple algebra shows that this expression equals

$$\tfrac{1}{4}(E(x_k, t^{n+1}) - E(x_k, t^n))(N(x_k, t^{n+1}) - N(x_k, t^n)),$$

and hence is $O(\Delta t^2)$.

As for σ_k^n, we use Taylor's theorem again to write

$$\begin{aligned}
\sigma_k^n &= (N_{tt}(x_k, t^n) + O(\Delta t^2)) - \frac{1}{2}(N_{xx}(x_k, t^{n+1}) + O(\Delta x^2)) \\
&\quad - \frac{1}{2}(N_{xx}(x_k, t^{n-1}) + O(\Delta x^2)) - \left(\frac{\partial^2}{\partial x^2}|E(x_k, t^n)|^2 + O(\Delta x^2)\right).
\end{aligned}$$

The result follows from (ZS.N), since

$$N_{xx}(x_k, t^n) - \tfrac{1}{2}(N_{xx}(x_k, t^{n+1}) + N_{xx}(x_k, t^{n-1})) = O(\Delta t^2). \quad \square$$

Recall that the errors are defined by (29), (30). In order to obtain the error equations we subtract (8.E) from the definition (34) of τ_k^n to get

$$\begin{aligned}
&i\left(\frac{e_k^{n+1} - e_k^n}{\Delta t}\right) + \frac{1}{2}\delta^2 e_k^n + \frac{1}{2}\delta^2 e_k^{n+1} \\
&= \tau_k^n + \frac{1}{4}[N(x_k, t^n) + N(x_k, t^{n+1})][E(x_k, t^n) + E(x_k, t^{n+1})] \\
&\quad - \frac{1}{4}[N_k^n + N_k^{n+1}][E_k^n + E_k^{n+1}] \\
&= \tau_k^n + \frac{1}{4}[(\eta_k^n + \eta_k^{n+1})(E(x_k, t^n) + E(x_k, t^{n+1})) \\
&\qquad + (N_k^n + N_k^{n+1})(e_k^n + e_k^{n+1})].
\end{aligned} \tag{36}$$

Subtracting $(8.N)$ from (35), the definition of σ_k^n, we get similarly

$$\frac{\eta_k^{n+1} - 2\eta_k^n + \eta_k^{n-1}}{\Delta t^2} - \frac{1}{2}\delta^2 \eta_k^{n+1} - \frac{1}{2}\delta^2 \eta_k^{n-1} = \sigma_k^n + \delta^2\left(|E(x_k, t^n)|^2 - |E_k^n|^2\right). \tag{37}$$

In a sequence of lemmas we will derive energy estimates on e and η.

Lemma 6 (L^2-estimate of e). *There are constants* c, c_T *such that for* $\Delta x, \Delta t$ *sufficiently small,*

$$\|e^{n+1}\|_2^2 \le (1 + c\Delta t)\|e^n\|_2^2 + c_T(\Delta t^2 + \Delta x^2)^2 \Delta t + c\Delta t\left(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2\right).$$

Proof. As in Theorem 1(a), we multiply (36) by $\bar{e}_k^{n+1} + \bar{e}_k^n$, sum on k, and take the imaginary part to get

$$\mathrm{I} + \mathrm{II} = \mathrm{III} + \mathrm{IV}, \tag{38}$$

where

$$\mathrm{I} = \frac{1}{\Delta t} \operatorname{Re} \sum_k (e_k^{n+1} - e_k^n)(\bar{e}_k^{n+1} + \bar{e}_k^n) = \frac{1}{\Delta t} \sum_k (|e_k^{n+1}|^2 - |e_k^n|^2),$$

$$\mathrm{II} = \frac{1}{2} \operatorname{Im} \sum_k (\bar{e}_k^{n+1} + \bar{e}_k^n)(\delta^2 e_k^{n+1} + \delta^2 e_k^n),$$

$$\mathrm{III} = \operatorname{Im} \sum_k (\bar{e}_k^{n+1} + \bar{e}_k^n)\tau_k^n,$$

$$\mathrm{IV} = \frac{1}{4} \operatorname{Im} \sum_k (\bar{e}_k^{n+1} + \bar{e}_k^n)[(\eta_k^n + \eta_k^{n+1})(E(x_k, t^n) + E(x_k, t^{n+1}))],$$

the last simplifying since N is real. All sums are taken over indices k with $1 \le k \le J$.

Term I is as desired. For III, we have from Lemma 5

$$\begin{aligned}|\mathrm{III}| &\le c \sum_k (|e_k^{n+1}|^2 + |e_k^n|^2) + c \sum_k |\tau_k^n|^2 \\ &\le c\Delta x^{-1}(\|e^{n+1}\|_2^2 + \|e^n\|_2^2) + c_T(\Delta t^2 + \Delta x^2)^2 \cdot J,\end{aligned}$$

and IV is easily estimable by

$$\begin{aligned}|\mathrm{IV}| &\le c \sup_{x, t \le T} |E(x, t)| \cdot \sum_k \frac{(|e_k^{n+1}| + |e_k^n|)\Delta x^{1/2} \cdot (|\eta_k^{n+1}| + |\eta_k^n|)\Delta x^{1/2}}{\Delta x} \\ &\le c\Delta x^{-1}[\|e^{n+1}\|_2^2 + \|e^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2].\end{aligned}$$

As before, term II vanishes upon summation by parts. Now we multiply (38) by $\Delta t \Delta x$ and use the bounds derived above to get

$$\begin{aligned}\|e^{n+1}\|_2^2 \le \|e^n\|_2^2 &+ c\Delta t(\|e^{n+1}\|_2^2 + \|e^n\|_2^2) + c_T(\Delta t^2 + \Delta x^2)^2 \cdot J\Delta t\Delta x \\ &+ c\Delta t(\|e^{n+1}\|_2^2 + \|e^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2).\end{aligned} \tag{39}$$

Thus, we have

$$\begin{aligned}(1 - c\Delta t)\|e^{n+1}\|_2^2 \le (1 + c\Delta t)\|e^n\|_2^2 &+ c_T(\Delta t^2 + \Delta x^2)^2 \Delta t \\ &+ c\Delta t(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2),\end{aligned} \tag{40}$$

and the result follows. □

When estimating the energy, we will need bounds on the discrete potentials u_k^n from Lemma 2 and U_k^n from Lemma 4.

Lemma 7. *There is a constant* c *depending only on the data such that*

$$\sup_k |u_k^n| \le c.$$

Proof. We write, using the boundary condition $u_0^n = 0$,

$$|u_k^n| = \left|\sum_{j=1}^{k}(u_j^n - u_{j-1}^n)\right| = \left|\Delta x \sum_{j=1}^{k} \delta u_{j-1}^n\right| \le \|\delta u^n\|_2 (J\Delta x)^{1/2},$$

and this is bounded by Lemma 3 and the definition of J. □

Lemma 8. *Let* U_k^n *be defined as in Lemma* 4. *There is a constant* c *such that*

$$\sup_k |U_k^n| \le c(\mathcal{E}^n)^{1/2}.$$

Proof. The proof is the same as that of Lemma 7, but in the last step we use the definition of $\mathcal{E}^n$ from Theorem 2. □

Lemma 9 (Energy of e). *Let* $h = \Delta t = \Delta x$, *and define*

$$\mathrm{II}^n = \frac{1}{2}\operatorname{Re}\sum_k \left(E(x_k, t^n) + E(x_k, t^{n+1})\right)\left(\eta_k^{n+1} + \eta_k^n\right)\bar{e}_k^{n+1},$$

$$\mathrm{III}^n = \frac{1}{2}\sum_k (N_k^{n+1} + N_k^n)|e_k^{n+1}|^2.$$

Then

$$\tfrac{1}{2}\|\delta e^n\|_2^2 + h(\mathrm{II}^{n-1} + \mathrm{III}^{n-1}) - (\tfrac{1}{2}\|\delta e^{n+1}\|_2^2 + h(\mathrm{II}^n + \mathrm{III}^n)) = O[h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^3].$$

Proof. As in Theorem 1(b), we multiply (36) by $(\bar{e}_k^{n+1} - \bar{e}_k^n)$, sum over k, $k = 1, \dots, J$, add the result to its conjugate, and take the real part. There results the identity

$$\mathrm{I}_0 = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\mathrm{I}_0 = \operatorname{Re}\sum_k (\bar{e}_k^{n+1} - \bar{e}_k^n)(\delta^2 e_k^n + \delta^2 e_k^{n+1}),$$

$$\begin{aligned}|\mathrm{I}| &= \left|2\operatorname{Re}\sum_k \tau_k^n(\bar{e}_k^{n+1} - \bar{e}_k^n)\right| \\ &\le c_T h^2 J^{1/2} h^{-1/2}(\|e^{n+1}\|_2 + \|e^n\|_2) \le c_T h(\mathcal{E}^n + \mathcal{E}^{n-1})^{1/2},\end{aligned}$$

$$\mathrm{II} = \frac{1}{2}\operatorname{Re}\sum_k (\eta_k^{n+1} + \eta_k^n)\left(E(x_k, t^n) + E(x_k, t^{n+1})\right)(\bar{e}_k^{n+1} - \bar{e}_k^n),$$

$$\mathrm{III} = \frac{1}{2}\sum_k (N_k^n + N_k^{n+1})(|e_k^{n+1}|^2 - |e_k^n|^2).$$

We sum I_0 by parts to get

$$\mathrm{I}_0 = \frac{1}{2}\sum_k |\delta e_k^n|^2 - \frac{1}{2}\sum_k |\delta e_k^{n+1}|^2. \tag{41}$$

Next, we rewrite term III as

$$\begin{aligned}\mathrm{III} &= \frac{1}{2}\sum_k [(N_k^{n+1}+N_k^n)|e_k^{n+1}|^2 - (N_k^n + N_k^{n-1})|e_k^n|^2 + (N_k^{n-1} - N_k^{n+1})|e_k^n|^2] \\ &\equiv \mathrm{III}^n - \mathrm{III}^{n-1} + \frac{1}{2}\sum_k (N_k^{n-1} - N_k^{n+1})|e_k^n|^2,\end{aligned}$$

where

$$\mathrm{III}^n = \tfrac{1}{2}\sum_k (N_k^{n+1} + N_k^n)|e_k^{n+1}|^2. \tag{42}$$

Recall from the definition (Lemma 2) of u_k^n that

$$\delta^2 u_k^n = \frac{N_k^{n+1} - N_k^n}{h}.$$

Thus,

$$\delta^2(u_k^n + u_k^{n-1}) = \frac{N_k^{n+1} - N_k^{n-1}}{h},$$

and therefore

$$\mathrm{III} = \mathrm{III}^n - \mathrm{III}^{n-1} - \frac{1}{2}h\sum_k |e_k^n|^2 \delta^2(u_k^n + u_k^{n-1}).$$

We sum by parts to get for the last term the bound

$$\begin{aligned} O\left(h\sum_k |e_k^n||\delta e_k^n|(|\delta u_k^n| + |\delta u_k^{n-1}|)\right) &= O(\|e^n\|_\infty \|\delta e^n\|_2(\|\delta u^n\|_2 + \|\delta u^{n-1}\|_2)) \\ &= O(\|e^n\|_2^{1/2}\|\delta e^n\|_2^{3/2}),\end{aligned}$$

where we have used Lemma 3. Hence,

$$\mathrm{III} = \mathrm{III}^n - \mathrm{III}^{n-1} + O(\mathscr{E}^{n-1}). \tag{43}$$

Consider now term II. For brevity we set

$$w_k^n = E(x_k, t^n) + E(x_k, t^{n+1}), \tag{44}$$

so that

$$w_k^n - w_k^{n-1} = E(x_k, t^{n+1}) - E(x_k, t^{n-1}) = O(h).$$

We write term II as

$$\begin{aligned}\mathrm{II} &= \tfrac{1}{2}\mathrm{Re}\sum_k (\eta_k^{n+1} + \eta_k^n) w_k^n (\bar{e}_k^{n+1} - \bar{e}_k^n) \\ &= \tfrac{1}{2}\mathrm{Re}\sum_k w_k^n \eta_k^{n+1}\bar{e}_k^{n+1} - \tfrac{1}{2}\mathrm{Re}\sum_k w_k^{n-1}\eta_k^n \bar{e}_k^n - \tfrac{1}{2}\mathrm{Re}\sum_k (w_k^n - w_k^{n-1})\eta_k^n \bar{e}_k^n \\ &\quad + \tfrac{1}{2}\mathrm{Re}\sum_k w_k^n \eta_k^n \bar{e}_k^{n+1} - \tfrac{1}{2}\mathrm{Re}\sum_k w_k^n \eta_k^{n+1}\bar{e}_k^n.\end{aligned}$$

Now we add and subtract the expression

$$\tfrac{1}{2}\operatorname{Re}\sum_k w_k^{n-1}\eta_k^{n-1}\bar{e}_k^n$$

and define

$$\mathrm{II}^n = \tfrac{1}{2}\operatorname{Re}\sum_k w_k^n \eta_k^{n+1}\bar{e}_k^{n+1} + \tfrac{1}{2}\operatorname{Re}\sum_k w_k^n \eta_k^n \bar{e}_k^{n+1}. \tag{45}$$

Then, using Lemma 4, we can write II as

$$\begin{aligned}\mathrm{II} &= \mathrm{II}^n - \mathrm{II}^{n-1} + O(\mathscr{E}^{n-1}) \\ &\quad - \tfrac{1}{2}\operatorname{Re}\sum_k \bar{e}_k^n[(w_k^n - w_k^{n-1})\eta_k^{n+1} + w_k^{n-1}(\eta_k^{n+1} - \eta_k^{n-1})] \\ &= \mathrm{II}^n - \mathrm{II}^{n-1} + O(\mathscr{E}^{n-1}) + O((\mathscr{E}^{n-1})^{1/2}(\mathscr{E}^n)^{1/2}) \\ &\quad - \tfrac{1}{2}\operatorname{Re}\sum_k h\bar{e}_k^n w_k^{n-1}\delta^2(U_k^n + U_k^{n-1}).\end{aligned} \tag{46}$$

We sum the last term here once by parts; it equals

$$\begin{aligned}&\tfrac{1}{2}\operatorname{Re}\sum_k h\delta(U_k^n + U_k^{n-1})(w_k^{n-1}\delta\bar{e}_k^n + \bar{e}_{k+1}^n\delta w_k^{n-1}) \\ &= O[(\|\delta U^n\|_2 + \|\delta U^{n-1}\|_2)(\|E(t^{n-1})\|_\infty\|\delta e^n\|_2 + \|E_x(t^{n-1})\|_\infty\|e^n\|_2)] \\ &= O((\mathscr{E}^n + \mathscr{E}^{n-1})).\end{aligned}$$

Using these estimates in (46), we have

$$\mathrm{II} = \mathrm{II}^n - \mathrm{II}^{n-1} + O[\mathscr{E}^n + \mathscr{E}^{n-1}]. \tag{47}$$

Finally, we multiply the relation

$$\mathrm{I}_0 = \mathrm{I} + \mathrm{II} + \mathrm{III}$$

by h and use the estimates for each of these terms derived above to get

$$\begin{aligned}\tfrac{1}{2}\|\delta e^n\|_2^2 - \tfrac{1}{2}\|\delta e^{n+1}\|_2^2 &= O(h^3) + O[h(\mathscr{E}^n + \mathscr{E}^{n-1})] \\ &\quad + \mathrm{II}^n h + \mathrm{III}^n h - \mathrm{II}^{n-1}h - \mathrm{III}^{n-1}h,\end{aligned} \tag{48}$$

or

$$\begin{aligned}&\tfrac{1}{2}\|\delta e^n\|_2^2 + h(\mathrm{II}^{n-1} + \mathrm{III}^{n-1}) - (\tfrac{1}{2}\|\delta e^{n+1}\|_2^2 + h(\mathrm{II}^n + \mathrm{III}^n)) \\ &\qquad = O(h(\mathscr{E}^n + \mathscr{E}^{n-1}) + h^3),\end{aligned} \tag{49}$$

and this is the statement of Lemma 9. □

Lemma 10 (η-energy). *Let* $h = \Delta t = \Delta x$. *Then*

$$\begin{aligned}&-\tfrac{1}{2}\|\delta U^n\|_2^2 - \tfrac{1}{4}(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \tfrac{1}{2}\|\delta U^{n-1}\|_2^2 + \tfrac{1}{4}(\|\eta^n\|_2^2 + \|\eta^{n-1}\|_2^2) \\ &\quad = O(h^5 + h(\mathscr{E}^n + \mathscr{E}^{n-1})).\end{aligned}$$

Proof. Recall from Lemma 4 the relation

$$\delta^2 U_k^n = \frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{h^2} = \frac{\eta_k^{n+1} - \eta_k^n}{h}.$$

We multiply the η-equation (37) by $\frac{1}{2}(U_k^n + U_k^{n-1})$ and sum over k to get the identity

$$I_1 - I_2 - I_3 = I_4 + I_5, \tag{50}$$

where

$$I_1 = \frac{1}{2}\sum_k (U_k^n + U_k^{n-1})\frac{(\eta_k^{n+1} - 2\eta_k^n + \eta_k^{n-1})}{h^2},$$

$$I_2 = \frac{1}{4}\sum_k (U_k^n + U_k^{n-1})\delta^2\eta_k^{n-1},$$

$$I_3 = \frac{1}{4}\sum_k (U_k^n + U_k^{n-1})\delta^2\eta_k^{n+1},$$

$$I_4 = \frac{1}{2}\sum_k \sigma_k^n(U_k^n + U_k^{n-1}) = O(h(\mathcal{E}^n + \mathcal{E}^{n-1})^{1/2}) \quad \text{(by Lemma 8)},$$

$$I_5 = \frac{1}{2}\sum_k (U_k^n + U_k^{n-1})\delta^2\{|E(x_k, t^n)|^2 - |E_k^n|^2\}.$$

We sum $I_2 + I_3$ by parts, with the result

$$I_2 + I_3 = -\frac{1}{4}\sum_k \delta(\eta_k^{n+1} + \eta_k^{n-1})\delta(U_k^n + U_k^{n-1}). \tag{51}$$

Expansion of this yields

$$\begin{aligned}
&-\frac{1}{4h^2}\sum_k (\eta_{k+1}^{n+1} + \eta_{k+1}^{n-1} - \eta_k^{n+1} - \eta_k^{n-1})(U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1}) \\
&\quad = -\frac{1}{4h^2}\sum_k (\eta_k^{n+1} + \eta_k^{n-1})(U_k^n + U_k^{n-1} - U_{k-1}^n - U_{k-1}^{n-1}) \\
&\quad\quad + \frac{1}{4h^2}\sum_k (\eta_k^{n+1} + \eta_k^{n-1})(U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1}),
\end{aligned}$$

where we put $k \to k-1$ to get the first sum. Thus,

$$\begin{aligned}
I_2 + I_3 &= \frac{1}{4h^2}\sum_k (\eta_k^{n+1} + \eta_k^{n-1})[U_{k+1}^n - 2U_k^n + U_{k-1}^n + U_{k+1}^{n-1} - 2U_k^{n-1} + U_{k-1}^{n-1}] \\
&= \frac{1}{4}\sum_k (\eta_k^{n+1} + \eta_k^{n-1})[\delta^2 U_k^n + \delta^2 U_k^{n-1}] \\
&= \frac{1}{4h}\sum_k (\eta_k^{n+1} + \eta_k^{n-1})[(\eta_k^{n+1} - \eta_k^n) + (\eta_k^n - \eta_k^{n-1})] \\
&= \frac{1}{4h}\sum_k ((\eta_k^{n+1})^2 - (\eta_k^{n-1})^2) \\
&= \frac{1}{4h}\sum_k ((\eta_k^{n+1})^2 + (\eta_k^n)^2) - \frac{1}{4h}\sum_k ((\eta_k^n)^2 + (\eta_k^{n-1})^2).
\end{aligned}$$

Term I_5 is summed once by parts, with the result

$$I_5 = -\frac{1}{2h^2}\sum_k \left(U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1}\right) \cdot \left(|E(x_{k+1}, t^n)|^2 - |E_{k+1}^n|^2 - |E(x_k, t^n)|^2 + |E_k^n|^2\right), \tag{52}$$

and further expansion yields

$$\begin{aligned} I_5 &= -\frac{1}{2h}\mathrm{Re}\sum_k \left(\delta U_k^n + \delta U_k^{n-1}\right) \\ &\qquad \cdot \left[\left(E(x_{k+1}, t^n) - E_{k+1}^n\right)\left(\overline{E}(x_{k+1}, t^n) + \overline{E}_{k+1}^n\right) - \left(E(x_k, t^n) - E_k^n\right)\left(\overline{E}(x_k, t^n) + \overline{E}_k^n\right)\right] \\ &= -\frac{1}{2h}\mathrm{Re}\sum_k \left(\delta U_k^n + \delta U_k^{n-1}\right) \\ &\qquad \cdot \left[e_{k+1}^n\left(\overline{E}(x_{k+1}, t^n) + \overline{E}_{k+1}^n\right) - e_k^n\left(\overline{E}(x_k, t^n) + \overline{E}_k^n\right)\right] \\ &= -\frac{1}{2h}\mathrm{Re}\sum_k (\delta U_k^n + \delta U_k^{n-1}) \Big[(e_{k+1}^n - e_k^n)\left(\overline{E}(x_{k+1}, t^n) + \overline{E}_{k+1}^n\right) \\ &\qquad + e_k^n\left(\overline{E}(x_{k+1}, t^n) - \overline{E}(x_k, t^n) + \overline{E}_{k+1}^n - \overline{E}_k^n\right)\Big] \qquad (53) \\ &= O\left(\sum_k (|\delta U_k^n| + |\delta U_k^{n-1}|)(|\delta e_k^n| + |e_k^n|(c_T + |\delta E_k^n|))\right) \\ &= O\left(h^{-1}(\mathscr{E}^n + \mathscr{E}^{n-1}) + h^{-1}\|e^n\|_\infty(\mathscr{E}^n + \mathscr{E}^{n-1})^{1/2}\|\delta E^n\|_2\right) \\ &= O\left(h^{-1}(\mathscr{E}^n + \mathscr{E}^{n-1})\right) \end{aligned}$$

by the Sobolev inequality applied to $\|e^n\|_\infty$.

Lastly, for the term I_1 we note from (31) that

$$\delta^2 U_k^n - \delta^2 U_k^{n-1} = \frac{1}{h}\left(\eta_k^{n+1} - \eta_k^n - (\eta_k^n - \eta_k^{n-1})\right) = \frac{\eta_k^{n+1} - 2\eta_k^n + \eta_k^{n-1}}{h},$$

and hence

$$I_1 = \frac{1}{2h}\sum_k (U_k^n + U_k^{n-1})\delta^2(U_k^n - U_k^{n-1}). \tag{54}$$

Summing by parts we get

$$I_1 = -\frac{h^{-2}}{2h}\sum_k \left[U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1}\right] \cdot \left[U_{k+1}^n - U_{k+1}^{n-1} - (U_k^n - U_k^{n-1})\right]. \tag{55}$$

This can be rewritten as

$$I_1 = -\frac{1}{2h}\sum_k \left[(\delta U_k^n)^2 - (\delta U_k^{n-1})^2\right] = -\frac{1}{2h^2}\left[\|\delta U^n\|_2^2 - \|\delta U^{n-1}\|_2^2\right]. \tag{56}$$

Returning now to (50), we multiply it by h^2 to get

$$\begin{aligned} &-\tfrac{1}{2}\|\delta U^n\|_2^2 + \tfrac{1}{2}\|\delta U^{n-1}\|_2^2 \\ &\quad -\tfrac{1}{4}(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \tfrac{1}{4}(\|\eta^n\|_2^2 + \|\eta^{n-1}\|_2^2) \\ &\quad = O\left(h(\mathscr{E}^n + \mathscr{E}^{n-1}) + h^5\right). \end{aligned} \tag{57}$$

This completes the proof. □

Proof of Theorem 2. Let us define $h = \Delta t = \Delta x$ and

$$H^{n-1} = \tfrac{1}{2}\|\delta e^n\|_2^2 + \tfrac{1}{2}\|\delta U^{n-1}\|_2^2 + \tfrac{1}{4}(\|\eta^n\|_2^2 + \|\eta^{n-1}\|_2^2). \tag{58}$$

Recall the definitions of the terms II^n, III^n from Lemma 9. Adding the conclusions of Lemmas 9 and 10, we get

$$\begin{aligned} H^n + h(\mathrm{II}^n + \mathrm{III}^n) = H^{n-1} &+ h(\mathrm{II}^{n-1} + \mathrm{III}^{n-1}) \\ &+ O(h(\mathscr{E}^n + \mathscr{E}^{n-1}) + h^3), \end{aligned} \tag{59}$$

where, from (33),

$$\mathscr{E}^n = \tfrac{1}{2}\|e^{n+1}\|_2^2 + H^n. \tag{60}$$

Now, for a (large) positive constant γ (to be chosen below) set

$$\widehat{\mathscr{E}}^n \equiv \gamma\|e^{n+1}\|_2^2 + H^n + h(\mathrm{II}^n + \mathrm{III}^n). \tag{61}$$

From (59) and Lemma 6 it follows that

$$\begin{aligned} \widehat{\mathscr{E}}^n \le{}& \gamma(1 + ch)\|e^n\|_2^2 + \gamma c_T h^5 + c\gamma h(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) \\ &+ H^{n-1} + h(\mathrm{II}^{n-1} + \mathrm{III}^{n-1}) + O(h(\mathscr{E}^n + \mathscr{E}^{n-1}) + h^3). \end{aligned} \tag{62}$$

Now we estimate II^n, III^n easily by

$$\begin{aligned} h|\mathrm{II}^n| &= \left|\frac{h}{2}\mathrm{Re}\sum_k (E(x_k, t^n) + E(x_k, t^{n+1}))(\eta_k^{n+1} + \eta_k^n)\bar{e}_k^{n+1}\right| \\ &\le c(\|E(t^n)\|_\infty + \|E(t^{n+1})\|_\infty)\|\eta^{n+1} + \eta^n\|_2\|e^{n+1}\|_2 \\ &\le \frac{1}{16}(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + c\|e^{n+1}\|_2^2 \end{aligned} \tag{63}$$

(with a constant c depending only on the data), and

$$\begin{aligned} h|\mathrm{III}^n| &\le \left|\frac{h}{2}\sum_k (N_k^{n+1} + N_k^n)|e_k^{n+1}|^2\right| \\ &\le c\|e^{n+1}\|_\infty\|N^{n+1} + N^n\|_2\|e^{n+1}\|_2 \\ &\le c(\|N^{n+1}\|_2 + \|N^n\|_2)\|e^{n+1}\|_2^{3/2}\|\delta e^{n+1}\|_2^{1/2} \end{aligned}$$

by the Sobolev inequality. Since the first factor is bounded by Lemma 3, we obtain

$$h|\mathrm{III}^n| \le \tfrac{1}{8}\|\delta e^{n+1}\|_2^2 + c\|e^{n+1}\|_2^2 \tag{64}$$

with c depending only on the data. Adding (63) to (64), we obtain

$$\begin{aligned} h(|\mathrm{II}^n| + |\mathrm{III}^n|) &\le \tfrac{1}{8}\|\delta e^{n+1}\|_2^2 + \tfrac{1}{16}(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + c\|e^{n+1}\|_2^2 \\ &\le \tfrac{1}{4}H^n + c\|e^{n+1}\|_2^2 \end{aligned} \tag{65}$$

by the definition (58) of H^n. It follows that $\widehat{\mathscr{E}}^n$ is strictly positive for a sufficiently large choice of γ, depending only on the data.

In fact, we can choose γ large enough so that $\gamma > 1$ and

$$\widehat{\mathscr{E}}^n \ge \frac{c}{2}\|e^{n+1}\|_2^2 + \frac{3}{4}H^n \tag{66}$$

with a constant $c > 0$ depending only on the data and on γ.

Hence, from (62),

$$\widehat{\mathscr{E}}^n \le \widehat{\mathscr{E}}^{n-1} + c_T\gamma h(\mathscr{E}^n + \mathscr{E}^{n-1}) + c_T\gamma h^3. \tag{67}$$

Now from its definition, we have, since $\gamma > 1$,

$$\begin{aligned}\mathscr{E}^n &= \tfrac{1}{2}\|e^{n+1}\|_2^2 + H^n < \gamma\|e^{n+1}\|_2^2 + H^n \\ &= \widehat{\mathscr{E}}^n - h(\mathrm{II}^n + \mathrm{III}^n) \le \widehat{\mathscr{E}}^n + \tfrac{1}{4}H^n + c\|e^{n+1}\|_2^2,\end{aligned} \tag{68}$$

where we have used (65). Since $H^n \le \mathscr{E}^n$ by (60), we conclude that

$$\tfrac{3}{4}\mathscr{E}^n \le \widehat{\mathscr{E}}^n + c\|e^{n+1}\|_2^2 \le c_\gamma \widehat{\mathscr{E}}^n \tag{69}$$

in view of (66). For any such (fixed) choice of γ, we obtain from (67)

$$(1 - c_T h)\widehat{\mathscr{E}}^n \le (1 + c_T h)\widehat{\mathscr{E}}^{n-1} + c_T h^3.$$

It follows that for $h = \Delta t = \Delta x$ sufficiently small, depending only on T and the data, we have

$$\widehat{\mathscr{E}}^n \le c_T[\widehat{\mathscr{E}}^0 + h^2].$$

Since $(\widehat{\mathscr{E}}^n)^{1/2}$ is equivalent to $(\mathscr{E}^n)^{1/2}$, the first part of the proof is complete.

It remains to estimate $\mathscr{E}^0$. From (29), (30) and (9), (10) we have

$$e_k^0 = 0, \quad \eta_k^0 = 0, \quad \eta_k^1 = O(h^2).$$

Thus, $\|\eta^1\|_2^2 + \|\eta^0\|_2^2 = O(h^4)$. From Lemma 6 with $n = 0$, $\|e^1\|_2^2 = O(h^5)$, and hence

$$\|\delta e^1\|_2^2 = h^{-1}\sum_{k=1}^{J} |e_{k+1}^1 - e_k^1|^2 \le 4h^{-1}\sum_{k=1}^{J} |e_k^1|^2 = O(h^3).$$

Finally, we bound $\|\delta U_k^n\|_2$. We multiply the definition of U_k^n by U_k^n, sum over k, and then sum by parts to get

$$\|\delta U^0\|_2^2 = -\sum_{k=1}^{J} U_k^0(\eta_k^1 - \eta_k^0) = \sum_{k=1}^{J-1}\sum_{j=1}^{J-1} G(x_k, x_j)\eta_k^1\eta_j^1,$$

where we have used Lemma 4 again. Since G is continuous, it follows from general considerations (or from explicit computation, using $\eta_k^1 = O(h^2)$) that the last expression is $O(h^2)$, and this completes the proof. □

Bibliography

1. H. Added and S. Added, *Equations of Langmuir turbulence and nonlinear Schrödinger equation: Smoothness and approximation*, J. Funct. Anal. **79** (1988), 183–210.
2. M. Delfour, M. Fortin, and G. Payne, *Finite difference solution of a nonlinear Schrödinger equation*, J. Comput. Phys. **44** (1981), 277–288.
3. J. Gibbons, S. G. Thornhill, M. J. Wardrop, and D. Ter Harr, *On the theory of Langmuir solitons*, J. Plasma Phys. **17** (1977), 153–170.
4. R. Glassey and J. Schaeffer, *Convergence of a second-order scheme for semilinear hyperbolic equations in $2 + 1$ dimensions*, Math. Comp. **56** (1991), 87–106.
5. G. L. Payne, D. R. Nicholson, and R. M. Downie, *Numerical solution of the Zakharov equations*, J. Comput. Phys. **50** (1983), 482–498.

6. J. M. Sanz-Serna, *Methods for the numerical solution of the nonlinear Schrödinger equation*, Math. Comp. **43** (1984), 21–27.

7. S. Schochet and M. Weinstein, *The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence*, Comm. Math. Phys. **106** (1986), 569–580.

8. W. Strauss and L. Vazquez, *Numerical solution of a nonlinear Klein-Gordon equation*, J. Comput. Phys. **28** (1978), 271–278.

9. C. Sulem and P. L. Sulem, *Regularity properties for the equations of Langmuir turbulence*, C. R. Acad. Sci. Paris Sér. A Math. **289** (1979), 173–176.

10. C. Sulem, P. L. Sulem, and H. Frisch, *Tracing complex singularities with spectral methods*, J. Comput. Phys. **50** (1983), 138–161.

11. V. E. Zakharov, *Collapse of Langmuir waves*, Soviet Phys. JETP **35** (1972), 908–912.

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